Problem:

Consider a one dimensional time-independent Schrödinger equation for some arbitrary potential $V(x)$. Prove that if a solution $\psi(x)$ has the property that $\psi(x) \rightarrow 0$ as $x \rightarrow \pm\infty$, then the solution must be nondegenerate and therefore real, apart from a possible overall phase factor.

Solution:

Suppose that there exist a second function $\phi(x)$ which satisfies the same Schrödinger equation with the same energy E as ψ and is such that $\phi(x) \rightarrow 0$ as $x \rightarrow \pm\infty$. Then,

$$\psi''/\psi = -2m(E-V)/\hbar^2$$
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so we have

$$\psi''\phi - \phi''\psi = 0$$

or

$$\psi'\phi - \phi'\psi = \text{constant}.$$

The boundary condition at $x \rightarrow \pm\infty$ then gives

$$\psi'\phi - \phi'\psi = 0 \qquad \Longrightarrow \qquad \frac{\psi'}{\psi} = \frac{\phi'}{\phi}.$$

After integration, we have $\ln\,\psi = \ln\,\phi + \text{constant}$ or $\psi = \phi \times \text{constant}$. Therefore, ϕ and ψ show the same state. Hence the solution is not degenerate.

When $V(x)$ is a real function, ψ^* and ψ satisfy the same equation with the same energy and the same boundary condition as $\psi^*(x) \rightarrow 0$ as $x \rightarrow \infty$. Hence, $\psi^* = c\psi$ so $c^2 = 1$